# Rotational Dynamic Design Formulae:

## Basic Definitions:

Many people are confused by rotational dynamic design problems. This is because few people teaching this subject have done so in an application oriented environment. It is really quite simple. The first thing to understand is the relationship between horsepower (HP), torque (T), and speed (rpm):

$$T = HP * 5252 / rpm \quad :Eq\ 1$$

where T is in lb-ft. Also, 1 HP = 745.7 W – which helps when you have a motor rated in Watts. Beware of older motors rated in *metric horsepower*. Changes in definitions such as the metre and Newton make a *metric horsepower* 1.0139 as large as the conventional (550 lb-ft/sec) definition of horsepower. This is rarely a real problem as other factors introduce larger errors that must be accounted in precise applications. Capacitor-start electric motors may exceed their nominal power rating by as much as a factor of 3 for short periods. Therefore, when calculating a torque for an electric motor at start-up, this equation may be rewritten as:

$$T = f * HP * 5252 / rpm \quad :Eq\ 1A$$

where f is the start-up power factor.

## Rotational Inertia:

In simplified linear dynamics, mass is the inertia of the object. We can rarely make that simplification in rotational dynamics. In most texts, the rotational inertia of an object is identified as $\mathbf{WR^2}$ where W is the mass of the object in question (usually measured in lbm in American or Imperial units) and $R^2$ is the square of the radius of gyration of the object. In this monograph, I will use the annotation $\mathbf{Wk^2}$ instead noting that k is the traditional mechanical symbol for the radius of gyration. The units for $\mathbf{Wk^2}$ are lb-ft². The thing to be careful of is accounting for any system level accelerations that may affect the lbm to lbf conversion. In a 1g environment, 1 lbm = 1 lbf. We habitually use this to simplify our equations, but when the assumption fails, we hang ourselves by the short and curlies quite thoroughly.

In any coaxial rotational system, the total $Wk^2$ for the system is the arithmetic sum of the components. In the case of cylindrical objects, $W = \pi r^2 L\rho$ r is the radius in inches, L is the length in inches, and $\rho$ is the density in lb/in³; $k^2 = r^2/2$ where r is the radius measured in feet. Complicated sections may be broken down into simpler sections and added or subtracted as required.

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| Steel | .28 lb/in³ | Aluminum | .10 lb/in³ | Copper | .32 lb/in³ | W-Carbide | .56 lb/in³ |
| Titanium | .22 lb/in³ | Zinc | .24 lb/in³ | Acetate | .047 lb/in³ | Polyester | .065 lb/in³ |
| Rubber | .045 lb/in³ | PVC | .052 lb/in³ | Nylon | .041 lb/in³ | Acetal | .054 lb/in³ |
| Lead | .40 lb/in³ | E-Glass | .092 lb/in³ | E-Carbon | .072 lb/in³ | Paper | ~.03 lb/in³ |

**Typical Material Densities :Table I**

 

| Volumetric Inertia/inch ($in^3$-$ft^2$/in) * ρ (lb/$in^3$) * Length (in) = $Wk^2$ (lb-$ft^2$): | | | |
|---|---|---|---|
| Diameter (inches): | Volumetric Inertia/inch: | Diameter (inches): | Volumetric Inertia/inch: |
| 0-1/4 | 0.0000027 | 9 | 4.47 |
| 0-3/8 | 0.0000135 | 9-1/4 | 4.99 |
| 0-1/2 | 0.000043 | 9-1/2 | 5.55 |
| 0-3/4 | 0.00022 | 9-3/4 | 6.16 |
| 1 | 0.00068 | 10 | 6.82 |
| 1-1/4 | 0.0017 | 10-1/4 | 7.53 |
| 1-1/2 | 0.0035 | 10-1/2 | 8.29 |
| 1-3/4 | 0.0064 | 10-3/4 | 9.10 |
| 2 | 0.0109 | 11 | 9.98 |
| 2-1/4 | 0.017 | 11-1/4 | 10.92 |
| 2-1/2 | 0.027 | 11-1/2 | 11.92 |
| 2-3/4 | 0.039 | 11-3/4 | 13.00 |
| 3 | 0.055 | 12 | 14.14 |
| 3-1/4 | 0.076 | 12-1/4 | 15.35 |
| 3-1/2 | 0.102 | 12-1/2 | 16.64 |
| 3-3/4 | 0.13 | 12-3/4 | 18.02 |
| 4 | 0.17 | 13 | 19.47 |
| 4-1/4 | 0.22 | 13-1/2 | 22.65 |
| 4-1/2 | 0.28 | 14 | 26.19 |
| 4-3/4 | 0.35 | 14-1/2 | 30.14 |
| 5 | 0.43 | 15 | 34.51 |
| 5-1/4 | 0.52 | 16 | 44.68 |
| 5-1/2 | 0.62 | 17 | 56.94 |
| 5-3/4 | 0.75 | 18 | 71.57 |
| 6 | 0.88 | 19 | 88.85 |
| 6-1/4 | 1.04 | 20 | 109.1 |
| 6-1/2 | 1.22 | 21 | 132.6 |
| 6-3/4 | 1.42 | 22 | 159.7 |
| 7 | 1.64 | 23 | 190.8 |
| 7-1/4 | 1.88 | 24 | 226.2 |
| 7-1/2 | 2.16 | 25 | 266.3 |
| 7-3/4 | 2.46 | 26 | 311.6 |
| 8 | 2.79 | 27 | 362.3 |
| 8-1/4 | 3.16 | 28 | 419.1 |
| 8-1/2 | 3.56 | 29 | 482.2 |
| 8-3/4 | 4.00 | 30 | 552.2 |

**Tabulated Inertia Values :Table II**

If we have a steel shaft that is ¾ inch in diameter and 20 inches overall length, then the $Wk^2$ for that shaft would be (.28 lb/in$^3$)(.00022 in$^3$-ft$^2$/in)(20 in) = **.0012 lb-ft$^2$**. If we add a 4 inch OD X ¾ inch wide zinc pulley to that shaft, the added $Wk^2$ would be defined as (.24 lb/in$^3$)(.17 in$^3$-ft$^2$/in)(.75 in) = **.0306 lb-ft$^2$** with a hole to be subtracted (.24 lb/in$^3$) (.00022 in$^3$-ft$^2$/in)(.75 in) = **. 000040 lb-ft$^2$** for a total pulley inertia of .0306 lb-ft$^2$- .000040 lb-ft$^2$ = **.0306 lb-ft$^2$**. (Please note that removing the hole in the center had no practical effect on the inertia – something to remember when in a hurry.) The total $Wk^2$ for the shaft and pulley system is given by: $Wk^2$(total) = $Wk^2$(shaft) + $Wk^2$(pulley) = .0012 lb-ft$^2$ + .0306 lb-ft$^2$ = **.0318 lb-ft$^2$**.

## Reflected Inertia:

Rotational systems often tie together several shafts operating at different speeds. This is the basis for much of machine design. Each shaft system is considered independently when calculating its inertia, but the effect each inertially rotating system has on the input varies according to the square of the ratio of drive speeds (rpm's). Thus, the reflected inertia of a belt-driven shaft to the input will be given by:

$$Wk^2(\text{reflected}) = Wk^2\ (\text{shaft-rpm}/\text{input-rpm})^2 \quad :\text{Eq 2}$$

In multi-centric systems, the summation of reflected $Wk^2$ factors defines the total system inertia. Reflected linear inertias must often be accounted in the total inertia. Conveyor systems are the most common source of these reflected linear inertias. The mass of the conveyor belt and maximum load becomes W (lb) and the radius of the pulley driving the load is r (ft). The inertial load of the conveyor belt system (each pulley and shaft system is calculated separately) is then determined as **$Wr^2$**. This is then reflected back into the input system as:

$$Wk^2(\text{reflected}) = Wr^2\ (\text{conveyor-drive-rpm}/\text{input-rpm})^2 \quad :\text{Eq 2A}$$

The total $Wk^2$ inertia at the input is the sum of all the reflected $Wk^2$ of the system.

## Acceleration Time:

The average torque required to accelerate a load to a given speed in a given time or the time required to accelerate a load to a given speed based on a given torque are commonly used design datapoints. The basic equation for average acceleration to or from a stopped condition is:

$$T(\text{avg}) = Wk^2(\text{rpm})/(308t) \quad :\text{Eq 3}$$

where: T(avg) is the average torque in lb-ft, $Wk^2$ is the inertia in lb-ft$^2$, rpm is the desired or initial speed in rpm, and t is the time in seconds. This equation can be reformulated into:

$$t = Wk^2(\text{rpm})/(308T(\text{avg})) \quad :\text{Eq 3A}$$

to solve for time (seconds). The thing you must be mindful of is the nature of the applied torque. With capacitor-start motors, the maximum torque may be 300% of the average torque. Clutches and breaks should have a **minimum** of a 15% margin above the average torque (see manufacturer's datasheets for specific recommendations). When you are accelerating a load from one speed (rpm) to another, the combined equations are:

$$T(\text{avg}) = Wk^2(\text{rpm2}-\text{rpm1})/(308t) \quad :\text{Eq 3B and}$$

$$t = Wk^2(\text{rpm2}-\text{rpm1})/(308T(\text{avg})) \quad :\text{Eq 3C}$$

where rpm1 is the initial speed and rpm2 is the final speed. If T(avg) or t is negative, that only means that you are slowing the system down. Please note that the total $Wk^2$ being managed must be reflected to the point where it is being managed. This, if a brake is applied at an output shaft, the entire $Wk^2$ for the system must be reflected and summed to that points. Do not forget the inertia of the motor when doing such calculations! When you are calculating start-up torque or time, the $Wk^2$ of the motor can (usually) be ignored as that is (usually) accounted in the power and speed rating of the motor itself.

**System Energy & Power Dissipation:**

A rotating system has energy. That energy, measured in ft-lbs, is given by:

$$E = Wk^2(rpm/9.549)^2/64.4 \quad :Eq\ 4$$

where units are consistent as defined herein. This is a specific application of the standard physics equation: $E = mk^2\varpi^2/2$ refined to engineering units. The value 64.4 should be thought of a 2g and the value 9.549 is the conversion from rpm to radians/second.

When the energy of the system is *pulsed*, such in a clutch/brake system for intermittent motion, the power consumed by the clutch or brake is increased by the frequency of the pulses. This power flow is given by:

$$P = [Wk^2(rpm/9.549)^2/64.4] * F \quad :Eq\ 5$$

where F is the pulse frequency in cycles/minute and P is measured in ft-lb/min for a system with a where the system is brought to a halt each cycle. If the system oscillates between rpm1 and rpm2, then the power flow is given by:

$$P = [Wk^2((rpm1-rpm2)/9.549)^2/64.4] * F \quad :Eq\ 5A$$

This is the heat in ft-lb/min the brake or clutch must be able to dissipate. Check with the manufacturer's specifications to verify compliance. Remember, 1 HP = 550 ft-lb/sec = 33,000 ft-lb/min and 1 W = 44.25 ft-lb/min.

**Flywheel Applications:**

A flywheel allows rotational energy to be supplied to a system to either: smooth out input power pulses (often called a *balance wheel*) or apply a lower power to intermittent high power output operations (often called an *energy flywheel*). A *balance wheel* is usually designed to supply the lowest practical rotational inertia to allow better system control while an *energy flywheel* is usually designed to minimize system speed changes by maintaining a relatively high rotational inertia. The difference between the two flywheel applications lies with the factors being minimized and is a convenience of terminology rather than a true distinction of practice.

**Balance Wheel Applications:** The most common application of a *balance wheel* is to smooth out piston power strokes in engines. The start of a power stroke in an IC engine will apply a very large torque to the crankshaft. The applied torque will decrease as the piston is driven away from "top dead center" (TDC) ideally reaching zero at "bottom dead center" (BDC). The *balance wheel* is designed to average out this variation to a small value (usually about 10%) at some minimum rotational speed value (often 1000 or 1500 rpm) to prevent pulsing from adversely affecting handling. In the case of fixed-speed power plants, the system will be designed around the nominal system speed.

Let's examine the case of an 800cc 4-cylinder automobile engine. Each piston will apply 125 lb-ft at TDC that will decrease in proportion to the square root of the stroke to 8.5 lb-ft at BDC at

 

1250 rpm (this data comes from a 1964 Austin Mini-Cooper). As this takes place two times per revolution, the relative timing is (.5 /rev / (1250 rev/min / 60 sec/min) =) 0.024 seconds. The relative rotational speeds are: rpm2 = 1250 rpm and rpm1 = (.9 * 1250 =) 1125 rpm. As Equation 3C tells us: $t = Wk^2(rpm2-rpm1)/(308T(avg))$. We can rearrange Equation 3C as:

$$Wk^2 = (308T(avg)t)/(rpm2-rpm1) \quad \text{Eq 6A}$$

where T(avg) = average torque in lb-ft and t = time in seconds. T(avg) = 2*(T(TDC)-T(BDC))/3 = 2*(125 – 8.5)/3 = 77.7 lb-ft. Therefore:

$$Wk^2 = 308(77.7)(0.024)/(1250-1125) = 4.59 \text{ lb-ft}^2$$

This is the inertia desired to smooth out the power input for the engine in question. It is now a simple task to design a *balance wheel* that meets this requirement.

**Energy Flywheel Application:** An *energy flywheel* application is often used to make-up energy in some cyclical force application. A common example is a punch press where a specific amount of energy is applied periodically. Refer to a machine design manual or text for information on how to compute the force required for a specific task. The force will be applied over a specific distance to become the *energy of operation* in ft-lb of energy. The energy is usually applied using a cam or scotch-yoke driven by the flywheel over some fraction (usually about 40%) of the single-revolution time which we call the period ($p$) of operation measured in seconds. This gives us energy/time – the definition of power: which we prefer to deal with as ft-lb/min.

An *energy flywheel* is usually required to operate at a frequency ($f$ in cycles/sec). The optimum design of an *energy flywheel* is an iterative process. We know at the outset the frequency and period such that the energy of application must occur during the period and the system must restore to operational speed during the *restoration period* $r = f - p$. What makes this more complicated is that the motor system is also supplying energy to the stroke of the press. Thus, the energy of application is equal to the sum of the energies supplied by the flywheel plus the motor. Further, the power supplied by the motor must be able to return the flywheel to operational speed during the *restoration period*. These are applications of equations 1: $T = HP * 5252/rpm$, 4: $E = Wk^2(rpm/9.549)^2/64.4$, and 3B: $T(avg) = Wk^2(rpm2-rpm1)/(308t)$.

$$\text{Energy of Operation} = 550*HP/p + Wk^2(rpm2/9.549)^2/64.4 \quad \text{Eq: 6B}$$

where rpm2 is nominal shaft speed usually specified to provide the correct velocity of closure of the press. As energy is taken from the system, the shaft will slow down to rpm1. Refer to a machine design manual or text for information on how to compute the proper ratio of allowable shaft speeds. As a very general rule is that rpm1 is 60% of rpm2. There are many exceptions to this rule, but it is a reasonable place to start when outlining assumptions for such an application. Further, applying equation 3B, we can see that:

$$T(avg) = \{(2*HP*5252/(rpm2-rpm1)) + Wk^2(rpm2-rpm1)/(308)\}/p \quad \text{Eq: 6C}$$

Reapplying equation 3B to the *restoration period*, we get:

$$(120*HP*5252/(rpm2-rpm1))/r = Wk^2(rpm2-rpm1)/(308r) \quad \text{Eq: 6D}$$

Playing equations: 6B, 6C, and 6D against one another (spreadsheets are **wonderful**!), it is fairly simple to derive a set of optimum solutions to an application such as this. It is worth remembering that minor changes cyclical frequency timing can reap great benefits in an *energy flywheel* system's design. Detailed examples of this type of design may be found in: *Machinery's Handbook* and Shigley's *Standard Handbook of Machine Design*.

 

**The Radius of Gyration (k):**

The formal definition of the radius of gyration for a general section is: $k = (I/A)^{0.5}$ where I is the area-based moment of inertia (measured in $ft^4$) about the axis of rotation and A is the area ($ft^2$) resulting in a value measured in ft. The radius of gyration for a circular section revolved about its center is $k^2 = r^2/2$ or $k = (r^2/2)^{0.5} = r/(2^{0.5})$.

The radius of gyration of a rectangular section rotated about its center of mass is given by: $k = ((b^2 + h^2)/12)^{0.5}$ where b is the *base width* and h is the *height* of the rectangle in ft.

The radius of gyration of a square section rotated about its center of mass is given by: $k = s/6^{0.5}$ where s is the length of the side of the square measured in ft.

The radius of gyration of a regular polygon rotated about its center of mass is given by: $k = ((6R^2 - a^2)/12)^{0.5} = ((12r^2 + a^2)/24)^{0.5}$ where R is the radius of the circumscribed circle to the polygon (ft), r is the radius of the inscribed circle to the polygon (ft), and a is the length of the polygon's flat (ft).

Other values for the radius of gyration may be determined by calculation or found in texts on mechanical dynamics or in engineering handbooks.