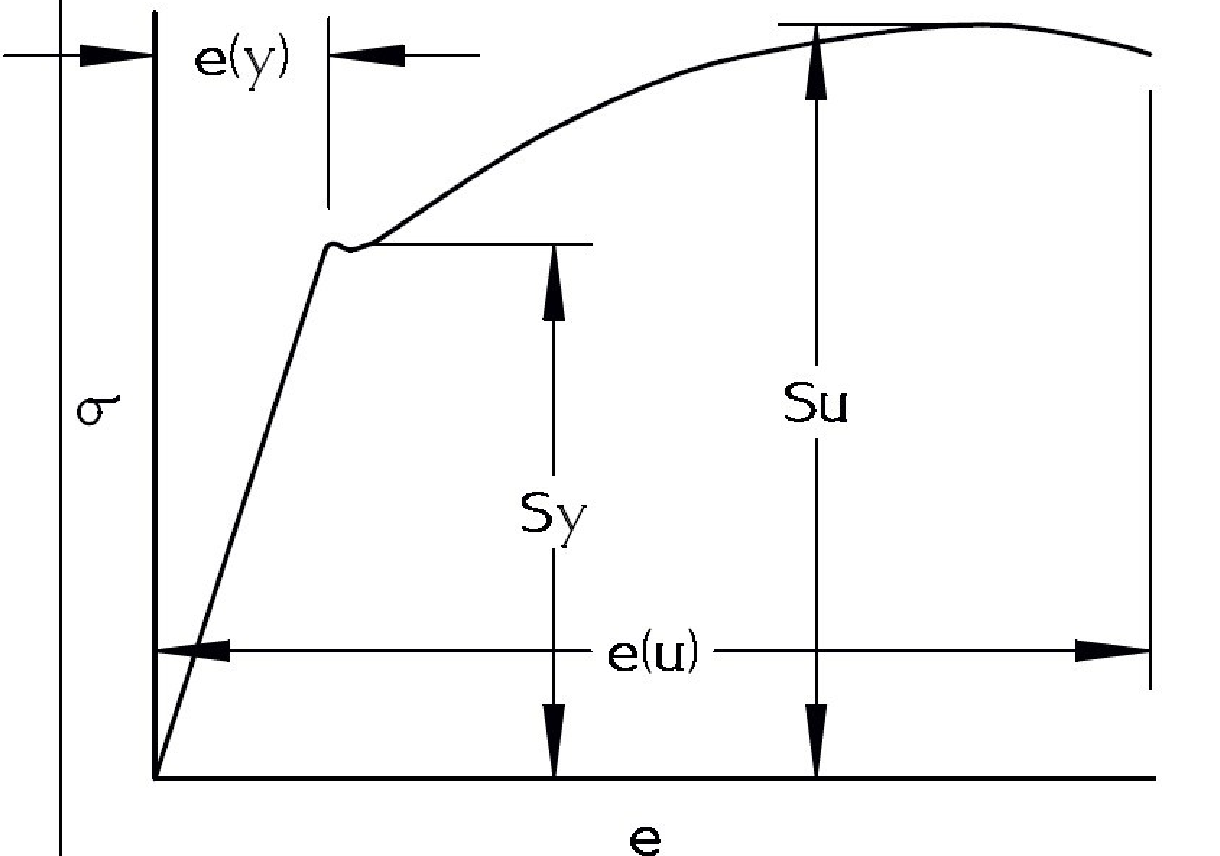
Introduction to Stress & Strain

When a load is applied to a material it deforms. The load is a **force** and is usually signified by the upper case letter **F** (for *force*) or **P** (for *pull*). We reduce this to a force applied to a **unit of area** (F/A) to make it consistent for comparison. This *Force/unit of area* is called a **stress** which is usually signified by the lower case Greek character *sigma* (**σ**) or the upper case character **S**. A stress applied to a material will cause it to deform consistently within broad limits. As the earliest concerns about this deformation dealt with elongation of the material, it is usually signified by the the lower case Greek character *epsilon* (**ε**) lower case letter **e**.



A Common Stress-Strain Diagram

Figure 1 shows a stress-strain diagram representative of many metals. The ***stress*** (***σ***) is the vertical axis and will be calibrated in units of Force/Area – most commonly lbs/in^2 (*psi*) or N/m^2 which is the metric unit the *Pascal* (*Pa*). The ***elongation*** (***e***) or ***strain*** is the horizontal axis and will be calibrated in units of Δ-Length/Length – commonly expressed as percentage (%), micro-inches/inch, or micro-meters/meter in stress-strain diagrams[1]. Because the units of length *cancel* in the expression, it is easy to forget that they are there.

The value ***Sy*** is the ***Yield Stress*** -- also called: *Yield Point*, or *Proportional Limit*. This is the load that when divided by the area being loaded, that will not permanently deform the material in question. As this value is hard to find, we actually allow the material to deform at this level of stress, the length of the sample (nominally 2.0000 inches long) is allowed to stretch by .2% (.0020 inches/ inch or .0040 inches total) when the load is removed. The value *e(y)* is the distance the sample will deform (measured in either length/length or percent) in the direction of the load that creates the stress. It is important to remember **which units of *elongation*** are in use. In inch-based units, *elongation* should be measured in *inches per inch*. In metric units, *elongation* should be measured in *meters/meter. Percent elongation* is often a convenient shorthand, but it needs to be carefully identified as such to avoid errors.

The portion of the *stress-strain* chart running from the origin (lower left corner) to the *Yield Point* is the ***region of elastic deformation*** for the material. The portion of the graph running from the *Yield Point* to the *Ultimate Break Point* is the ***region of plastic deformation***. These are terms common to *stress-strain* discussions that you should learn.

The slope of this linear portion of the *region of elastic deformation* is known as the ***Tensile Modulus*** (symbolized by the letter ***E***)[2]. It is the value *Sy/e(y)* which is usually expressed in *Lbs/in²* (or *psi*) in inch-based units or in *N/m²* (or *Pascals – Pa*) in metric units. The *Tensile Modulus* is given in material property tables. This is somewhat of a misnomer as the proper designation is more correctly stated as: *lbs/in²/in/in* or *N/m²/m/m*. This distinction may be pedantic, but it does help to remember the elongation factor in the *Tensile Modulus*. Thus, the *elongation* of a part may be found by dividing the applied *stress* (so long as it is less than the *Yield Stress*) by the *Tensile Modulus* (*E*) – so long as the units are consistent. The *elongation* at *Yield Stress* is easily calculated as: *e(y) = Sy/E* (because the value *e(y)* is rarely given in materials property tables).

Once a material starts yielding (i.e. it will not return to its original length), the curve ceases to be linear. Usually, but not always, the maximum applied stress or ***Ultimate Stress*** (***Su***) appears before the material actually fails (***Ultimate Strain*** or ***e(u)*** – also known as the ***Ultimate Break Point***). Properly determining how much *elongation* is created by a given *stress* in this area is non-trivial mathematics. The *Ultimate Strain* is given in material property tables as the percentage *Elongation at Failure*.

The area of the diagram immediately around the *Yield Point* is interesting. As the defined *Yield Stress* is surpassed, accumulated *elastic strain energy* is released into the surrounding structure of the material as *plastic strain* which begins to *slip* along rearranged planes – often associated with a *localized reduction in* area called *necking*. *Strain* will often (but not always) increase even with no additional *stress* because of this transformation. *Stress* will often decrease in the area in question because (A) the *slippage* redistributes the load, (B) the *area* being loaded decreases due to *necking* caused by the *slippage*, and (C) the tensile (or compression) testing machine has to "catch up" with the change in length (*strain*) caused by *slippage*. Metallurgy texts refer to this phenomena as *engineering stress* versus *true stress*. This is an advanced area of consideration in stress-strain analysis beyond the scope of this document. However, knowing that these factors exist is important even at this introductory level of discussion.

Also note that Figure 1 shows a decrease in stress happening after the *Ultimate Stress* has been reached but before the *Ultimate Break Point* has been reached. This is another critical stress point in a material where the form of its structure changes. This attribute is most common in ductile materials.

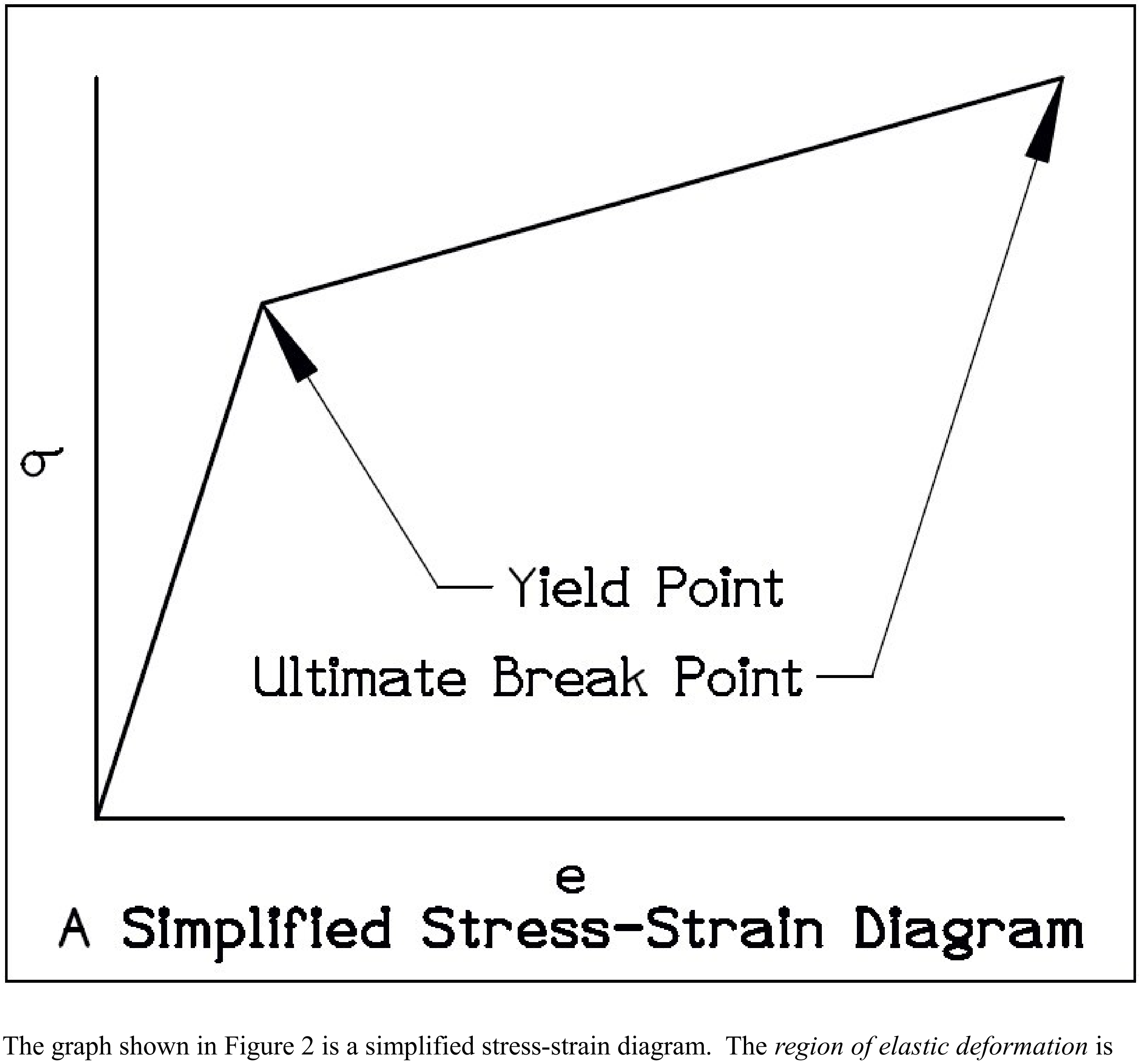
As a material is stretched or compressed, energy is expended reshaping the material's internal structure. This results in a change in the volume of the sample. The measure of that change in volume is *Poisson's Ratio* – signified by the lower case Greek letter *nu* (**ν**) – and is a characteristic property of a material. It is important to understand how this affects measurements of stress.

A perfectly elastic material will maintain the same volume no matter how stretched or compressed up to its failure (*Ultimate Break Point*). Natural rubber is such a material. It has a *Poisson's Ratio* $\nu = 0.5$. A perfectly non-elastic and non-brittle material will increase or decrease its volume by an amount equal to its cross-sectional area times the change in length. Certain grades of cork come very close to this ideal condition. They have a *Poisson's Ratio* $\nu = 0$. In most common cases, *Poisson's Ratio* is: $0 \leq \nu \leq 0.5$.

Although very uncommon, certain materials exist such that an increase in length by strain will cause an increase in their volume or a decrease in length by strain will cause a decrease in their volume greater than merely the cross-sectional area times the change in length. Such materials will have a *Poisson's Ratio* $\nu < 0$. The only natural occurring materials of this type are felts made from a small class of fibers. Composite materials with a negative *Poisson's Ratio* are being engineered today and will become less uncommon in the future. Be aware that they exist, but you are unlikely to run into them in practice.

The shape of the curve in the area between the *Yield Point* and *Ultimate Failure* is a power curve of the form: $e = Q - f(S^{(1/\nu)})$ – where Q is some parametric function, $f()$ is a derived function, S is the applied *stress*, and e is the elongation. This, as stated earlier, is obnoxiously complex mathematics.

There is a simplification we can use.



The graph shown in Figure 2 is a simplified stress-strain diagram. The *region of elastic deformation* is the same as a real *stress-strain* diagram. The difference lies in the *region of plastic deformation*. We make the simplification of using a straight line between the *Yield Point* to the (somewhat phony) point where *Ultimate Stress* and *Elongation at Failure* meet. The slope of this (somewhat phony) line is known as the ***Secant Modulus*** (symbolized as $\boldsymbol{M_s}$). It is **not accurate**, but it does provide an **estimate** of *elongation* versus *stress* in the *region of plastic deformation*\3.

The *Secant Modulus* for a material may be easily calculated. The values for: *Su*, *Sy*, *e(y)*, and *e(u)* in consistent units are needed. As noted above, the *elongation* at the *Yield Stress* may be found as: *e(y) = Sy/E*. Remember that this gives us a value in terms of Δ-Length/ Length. The value for *elongation* at the *Ultimate Break Point* is commonly available from material property tables – given in units of percent (%). To turn this into units of Δ-Length/Length, divide the percentage value by 100. Thus, the value for the *elongation* between the *Yield Point* and the *Ultimate Break Point* (*Δe(u-y)*) is found as: *Δe(u-y)* = ((*e(u)*/ 100) – *Sy*/*E*). The value for the change in *stress* from *Su* to *Sy* is trivial (*ΔS(u-y)* = *Su* – *Sy*). Thus:

$$\mathbf{M_s} = \Delta S(u\text{-}y)/\Delta e(u\text{-}y) = (Su - Sy)/((e(u)/100) - Sy/E) \leftarrow \text{Equation 1.}$$

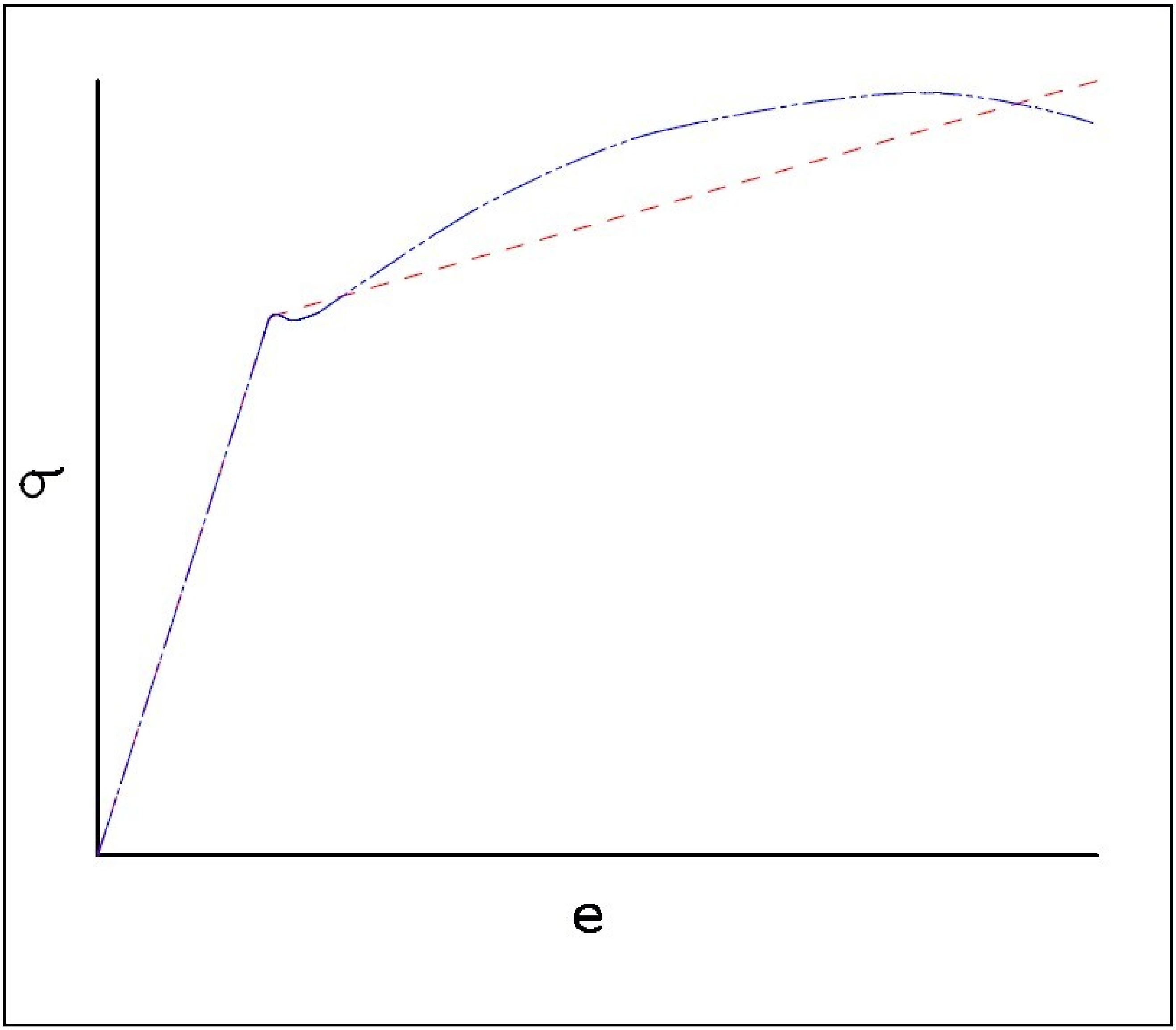


Figure 3 shows the relative error commonly found when using the *Secant Modulus*. The red dashed line is the simplified *stress-strain* diagram while the blue phantom line is the common *stress-strain* diagram. As noted above, both of these approaches are equally accurate in the *region of elastic deformation*. Using the *Secant Modulus* in the *region of plastic deformation* will typically show **greater** *strain* than more complicated analysis. If this is not acceptable, then use of a *Finite Element Analysis* (*FEA*) program with *large deformation analysis* should be used!

One thing to understand about the *Modulus of Elasticity* is that we have agreed to consider it a constant of a material. This is mostly sort-of kind-of true. Unless one goes to the expense and trouble of finding an exact value through production lot testing, the number is a nominal value. Treating it as a constant simplifies calculations without inserting significant error in our analysis. Very few designs are adversely affected by this assumption. Three significant digits is sufficient for a *Modulus* value.

Another thing to understand about the *Modulus of Elasticity* is that it relates stress to strain directly. So long as the stress is below the *Yield Point* of a material, the strain in that material will be given by:

$$\Delta = L * (\sigma/E) \leftarrow \text{Equation 2.}$$

where Δ is the change in length of the part and L is the original length of the part. A factor of -1 is used to account for compression rather than tension. The units for length, stress, and modulus must agree.

The *region of plastic deformation* is both less applicable to most designs and much less accurate a model for a material's behavior. The slope of the straight line shown in Figure 2 represents the *Secant Modulus*, represented by the symbol $\mathbf{M_s}$, used in simplistic deformation analysis. In this region, the sample is elongating (or foreshortening) such that it **will not** return to its original length. The simplified diagram is untrue. The strain varies as stress is applied in an observably non-linear manner. Truly accurate analysis of this condition is complex far beyond the scope of this paper. This was a commonly used simplification (i.e. S*ecant Modulus*) in the days before modern computers. It will give a quick estimate of deformation in this area even though it is known to be incorrect. As with the *Modulus of Elasticity*, the *Secant Modulus* provides us with the means to **estimate** the change in length seen when a material is stressed beyond the Yield Point. That estimation is given by:

$$\Delta = L * [(\sigma_y/E) + (\sigma - \sigma_y)/ M_s] \leftarrow \text{Equation 3.}$$

where the symbology agrees with previous usage. This is rarely (if ever) used in compression as there are factors not encompassed by this equation (notably *buckling*) that is beyond the scope of this paper. As with Equation 2, the units for length, stress, and modulus must agree.

The change in the cross-sectional area of a component under load is given as:

$$\Delta A = \Delta L*(1\text{-}2\ \nu) \leftarrow \text{Equation 4.}$$

Thus, in a simple round or square part, the initial volume may be described as: $V_i = A * L$. After undergoing a strain of ΔL, the final volume may be described as: $Vf = (L + \Delta L)*[A - \Delta L*(1\text{-}2\ \nu)]$. If we take an ASTM standard tensile test specimen of A36 steel, then we are dealing with values of: σ_y = 36,300 psi; E = 29,000,000 psi; and ν = .260. The test specimen will be: ø.500 inches X 2.000 inches long for an initial volume of .3927 in^3. At yield stress, that is an elongation of: $e = L * (\sigma/E)$ = 2.000 * (36,300/ 29,000,000) = .0025 inches. The initial cross-section area is: $A_i = \pi * (.500/2)^2 = .1963$ in^2. The reduction in area caused by this stress is, using Equation 4, $\Delta A = \Delta L*(1\text{-}2\ \nu)$ = .0025*(1-2*.260) = .0012 in^2. Thus, the area of the final cross-section is: $A_f = A_i - \Delta A$ = .1963 - .0012 = .1951 in^2. As the final length is (2.000 + .0025 =) 2.0025 inches, the final volume is (2.0025 * .1951 =) .3907 in^3 – 98% of the original volume.

This tells us that, in the *region of elastic deformation* for a material, changes in area or volume are rarely significant until the ratio of length to cross-sectional area (L/A), often called the *slenderness ratio*, is greater than 25 – though it is not uncommon to use values of 50 or 100 as critical values for a slenderness ratio in certain applications. The *slenderness ratio* in the above example is (2/.1963 =) 10.2. It is really only when working in the *region of plastic deformation* that such considerations become important in typical design.

Another area of importance for *Poisson's Ratio* is relating the various modulus values. We have already introduced the *Modulus of Elasticity* (E) which may be defined as $E = \sigma/e$ within the elastic range. It relates axial stress to longitudinal deformation. However, there is also the *Shear Modulus* (**G**) which relates longitudinal stress to axial deformation or torsional stress to radial deformation (i.e. twist). The *Shear Modulus* may be given with respect to the *Modulus of Elasticity* as:

$$G = E/[2(1+\nu)] \leftarrow \text{Equation 5.}$$

Another modulus of note is the *Bulk Modulus* (**K**) which relates stress to changes in a material's volume. The *Bulk Modulus* may be given with respect to the *Modulus of Elasticity* as:

$$K = E/[3(1-\nu)] \leftarrow \text{Equation 6.}$$

The *Secant Modulus* is rarely given in material property tables. It is, as noted above, a very inaccurate tool. However, an inaccurate tool is better than no tool and it **does** allow a designer to make an educated guess when dealing with plastic deformation of a part. The values needed to determine a value for the *Secant Modulus* are a material's: *Ultimate Tensile Stress*, *Yield Stress, Tensile Modulus*, and *Elongation at Rupture*. The *ΔY* for this modulus is: $(\sigma_u - \sigma_y)$. The *ΔX* for this modulus is: $[(\text{Elong}\%/100) - (\sigma_y/E)]$. Thus, the *Secant Modulus* reduces to:

$$M_s = (\sigma_u - \sigma_y)/[(\text{Elong}\%/100) - (\sigma_y/E)] \leftarrow \text{Equation 7.}$$

Dividing the *Elongation at Rupture* (Elong%) value by 100 converts it from a percentage to either inches per inch (in/in) or meters per meter (m/m) values compatible with the *Tensile Modulus* value.

Another thing to keep in mind is that characteristic values depend on temperature, rate of loading, and, in many cases, post-processing treatments. Most commonly the published values for materials will be *standard temperature* (i.e. applied at 20°C or 68°F) values. If the design is to be used at temperatures far different from *standard temperature*, these values will need to be adjusted. If the rate of loading or any other aspect of the design's application is non-standard, these values will need to be adjusted – speak with your suppliers or look in recognized standards for such values.

Characterizing Materials:

Materials are generally classed as being: strong or weak, stiff or flexible, ductile or brittle, or heavy or light. These characteristic terms are often imprecise as multiple factors may impact their application in a design. Stiffness, for example, is generally perceived according to the *Tensile Modulus* of the material multiplied by the ***Area Moment of Inertia*** (symbolized by an upper case **I**) of the part in question (see the section on ***Beam Properties*** for more information on the *Area Moment of Inertia*). To assume that an alloy steel with E = 29,700,000 psi will necessarily be stiffer than 2024 aluminum with E = 10,600,000 psi can be a mistake. It may be possible to create a section in the aluminum with more than (29.7/10.6 =) 2.8X as great an *Area Moment of Inertia* which would give the stiffness advantage to the aluminum part. It is important to keep such things in mind when characterizing materials for a specific application.

There are three general measures of a material's strength: *Yield Stress* and *Ultimate Tensile Stress* have been introduced and ***Impact Strength*** is the ability of material to absorb energy prior to fracture. *Impact Strength* does not immediately enter into this monograph, but it should be introduced here. A material's *Impact Strength* is a characteristic of a material that is often more dependent on thermal and mechanical conditioning than many other aspects of a material's characteristics. It is usually measured in ft-lbs, N-m, or Joules.

In direct comparison (recognizing the comparisons noted immediately above), the three *Modulus* values (*Tensile*, *Shear*, and *Bulk*) are measures of a material's intrinsic stiffness. The greater the *Modulus* value, the stiffer the materials. In most general terms, a component's stiffness will be proportional to the product of E and I (as noted above). Shape plays a role as well as *Modulus* in a part's stiffness, but *Modulus* is extremely important.

Plasticity is the state where strain causes permanent deformation. ***Ductility*** is *plasticity* under tension. ***Malleability*** is *plasticity* under compression. *Elongation* is the measure of *plasticity*. The greater the total elongation of a material, the more *plastic* it is in deformation. Although not always true, the more *plastic* a material is, the greater its ability to dissapate energy – also called ***toughness***. The *Secant Modulus* is a general measure of *plasticity* and *toughness*. The lower the value of the *Secant Modulus*, the more energy is absorbed through *elongation*.

Density is the property of mass/unit-volume that, when multiplied times the volume of a part, determines its mass – and is symbolized by the lower case Greek character *rho* (**ρ**). It is most commonly expressed in units of: lbm/in^3, lbm/ft^3, g/cm^3, or kg/m^3. We have not dealt with it so far in this section as it is not normally part of the stress & strain analysis. However, it **is** a property the dutiful designer needs to keep in mind, so it is introduced here. Total mass often plays as important a role in the functioning of a design as stress or strain.

Some Tabulated Metals Property Values:

The following tables contain ***typical*** values for the cited metals. They are derived from commonly used public sources. Critical applications **require** the use of **minimum** values either from appropriate handbooks or from *certified material reports* issued by your supplier!

Aluminums:

Material:	Density (lbm/in^3):	Poisson's Ratio:	Yield Stress (ksi):	Ultimate Stress (ksi):	Elongation (%):	Tensile Modulus (10^6 psi):	Shear Modulus (10^6 psi):	Bulk Modulus (10^6 psi):	Secant Modulus (ksi):
AL 1100-O	.098	.330	2.9	13.0	15.0	10.0	3.77	9.8	67.5
AL 5052-H32	.097	.330	28.0	33.0	12.0	10.2	3.76	10.0	42.6
AL 5052-H34	.097	.330	31.0	38.0	16.0	10.2	3.76	10.0	44.6
AL 2024-T4	.100	.330	37.7	57.3	8.0	10.6	4.06	10.4	226.0
AL 2024-T6	.100	.330	45.7	60.2	5.0	10.6	4.06	10.4	256.0
AL 6061-T4	.098	.330	21.0	35.0	22.0	10.0	3.77	9.8	64.3
AL 6061-T6	.098	.330	40.0	45.0	17.0	10.0	3.77	9.8	30.1
AL 6063-T4	.098	.330	13.0	25.0	22.0	10.0	3.74	9.8	54.9
AL 6063-T6	.098	.330	31.0	35.0	15.0	10.0	3.74	9.8	33.7
AL 7075-T6	.102	.330	73.0	83.0	3.0	10.4	3.90	10.2	435.0

Brasses & Bronzes:

Material:	Density (lbm/in^3):	Poisson's Ratio:	Yield Stress (ksi):	Ultimate Stress (ksi):	Elongation (%):	Tensile Modulus (10^6 psi):	Shear Modulus (10^6 psi):	Bulk Modulus (10^6 psi):	Secant Modulus (ksi):
SAE 360	.307	.311	45.0	55.8	20.0	14.1	5.37	1.78	54.9
Cartridge	.308	.375	39.9	53.7	30.0	16.0	5.80	1.33	46.4
Red Brass	.316	.307	10.0	39.2	48.0	16.7	6.38	2.15	60.9
Arsenical	.301	.346	27.0	60.0	55.0	17.0	6.32	1.75	60.2
SAE 660	.323	.340	18.1	34.8	20.0	14.5	5.41	1.55	84.0

Material:	Density (lbm/in³):	Poisson's Ratio:	Yield Stress (ksi):	Ultimate Stress (ksi):	Elongation (%):	Tensile Modulus (10^6 psi):	Shear Modulus (10^6 psi):	Bulk Modulus (10^6 psi):	Secant Modulus (ksi):
SAE 507	.320	.340	19.0	47.0	64.0	16.0	5.95	1.71	43.8
SAE 841	.293	.320	11.0	14.0	1.0	14.5	5.50	1.74	324.0

Steels:

Material:	Density (lbm/in³):	Poisson's Ratio:	Yield Stress (ksi):	Ultimate Stress (ksi):	Elongation (%):	Tensile Modulus (10^6 psi):	Shear Modulus (10^6 psi):	Bulk Modulus (10^6 psi):	Secant Modulus (ksi):
AISI 1015	.284	.290	41.3	55.8	37.0	29.0	11.6	23.0	39.3
AISI 1020	.284	.290	29.7	55.1	25.0	29.0	11.6	23.0	102.0
AISI 1117	.284	.290	41.3	61.6	33.0	29.0	11.6	23.0	61.8
ASTM A36	.284	.290	36.3	58.0	20.0	29.0	11.6	20.0	109.0
ASTM A283	.284	.290	23.9	45.0	27.0	29.0	11.6	20.0	78.4
AISI 4130	.284	.290	63.1	97.2	25.5	29.7	11.6	23.6	135.0
AISI 4340	.284	.290	68.2	108.0	22.0	29.7	11.6	23.6	183.0
AISI 8620	.284	.290	53.7	76.9	31.0	29.7	11.6	23.6	68.5

Stainless Steels:

Material:	Density (lbm/in³):	Poisson's Ratio:	Yield Stress (ksi):	Ultimate Stress (ksi):	Elongation (%):	Tensile Modulus (10^6 psi):	Shear Modulus (10^6 psi):	Bulk Modulus (10^6 psi):	Secant Modulus (ksi):
301	.290	.270	30.0	75.0	40.0	30.6	12.0	22.2	113.0
301 1/4 Hard	.290	.270	75.0	125.0	25.0	28.7	11.3	20.8	202.0
301 1/2 Hard	.290	.270	110.0	150.0	18.0	27.9	11.0	20.2	227.0
301 3/4 Hard	.290	.270	125.0	175.0	12.0	27.3	10.8	19.8	433.0
301 Full Hard	.290	.270	140.0	185.0	9.0	28.4	11.2	20.6	529.0
303	.289	.250	34.8	89.9	50.0	28.0	11.2	22.2	110.0
316	.289	.290	36.3	81.9	55.0	28.0	12.5	22.2	83.1
316L	.289	.290	29.7	74.7	60.0	28.0	12.5	22.2	75.1

Notes:

1) When dealing with stress-strain diagrams, *elongation* is measured in units of either Δ-Length/Length or percent often called *engineering strain*. When dealing with experimental strain tests, *elongation* is measured in units of Δ-Length and is more properly called *absolute elongation* – though test engineers rarely make this distinction.

2) The *Tensile Modulus* is also called *Young's Modulus* (in honor of Thomas Young) or the *Elastic Modulus*. The term *Tensile Modulus* is used here as many material have a *Compressive Modulus* of a value different from the *Tensile Modulus*.

3) The *Secant Modulus* is normally considered obsolete today. Advances in computer aided analysis do a much better job of estimating *elongation* and *rupture* in parts and assemblies. However, it is still valid for finding a quick *estimate* of the stress-strain situation in the *region of plastic deformation*.

Generally:

I use the terms *Yield Stress* and *Ultimate Stress* rather than *Yield Strength* and *Ultimate Strength* for a reason. The values provided in *force/unit-area* define a ***stress***. A *strength* is a load value that should be measured as a ***force***. Whereas a 1 in² cross-sectional area bar of ASTM A36 steel should *begin to yield* at a ***load*** of 36,300 lbs, a 2 in² cross-section area bar of the same material should *begin to yield* at a ***load*** of 72,600 lbs. However, they **both** *begin to yield* at a ***stress*** of 36,300 lbs/in² (psi)!

